5 dual space

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Def. 10.1 Given a vector space E, the vector space $E^* = \mathrm{Hom}(E, K)$ of linear maps from E to the field K is called the **dual space (or dual)** of E.
The linear maps in E^* are called **linear forms**, or **covectors**.
The dual space E^{**} of the space E^* is the **bidual** of E

Notation: Linear forms $f: E \to K$ will be denoted with a star, e.g. u^*, x^*.

$\forall f^* \in E^*$, if $(u_1, \dots, u_n)$ is a basis of E, $\forall x = x_1 u_1 + \dots + x_n u_n \in E$,

$$f^*(x) = f^*(u_1)x_1 + \dots + f^*(u_n)x_n$$
$$\equiv c_1 x_1 + \dots + c_n x_n$$

Therefore, f^* can be represented by

$$f^*(x) = (c_1 \ \cdots \ c_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = c^T x .$$

Def. 10.2 Given a vector space E and any basis $(u_i)_{i \in I}$ for E, for every $i \in I$, there is a unique linear form u_i^* such that

$$u_i^*(u_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \quad \forall j \in I.$$ (Kronecker delta δ_{ij}.)

u_i^* is called the **coordinate form** of index i w.r.t. the basis $(u_i)_{i \in I}$.

It's called the coordinate form because if E is finite dimensional, $\forall v = \lambda_1 u_1 + \dots + \lambda_n u_n$, $u_i^*(v) = \lambda_i$.
So u_i^* is the linear function that returns the ith coordinate.

Thm 2.3/2.20 Let E be a vector space of dim n.
For every basis $(u_1, \dots, u_n)$ of E, the family of coordinate

Thm 2.3/2.20 Let E be a vector space of ... For every basis $(u_1, \ldots, u_n)$ of E, the family of coordinate forms $(u_1^*, \ldots, u_n^*)$ is a basis of E^*, called the **dual basis** of $(u_1, \ldots, u_n)$.

Aside: If $P = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}$, then $P^{-1} = \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix}$

Notation: Denote $u^*(v)$ by $\langle u^*, v \rangle$, a bilinear map $\langle -, - \rangle : E^* \times E \to K$ satisfying

$$\langle u_1^* + u_2^*, v \rangle = \langle u_1^*, v \rangle + \langle u_2^*, v \rangle$$
$$\langle u_1^*, v_1 + v_2 \rangle = \langle u^*, v_1 \rangle + \langle u^*, v_2 \rangle$$
$$\langle \lambda u^*, v \rangle = \lambda \langle u^*, v \rangle$$
$$\langle u^*, \lambda v \rangle = \lambda \langle u^*, v \rangle.$$

This bilinear map is the *canonical pairing* between E^* and E.

We also have the map $eval_v : E^* \to K$, $eval_v(u^*) = \langle u^*, v \rangle$.

Def. 10.3 $v \in E$ and $u^* \in E^*$ are *orthogonal* iff $\langle u^*, v \rangle = 0$.
Two subspaces V of E and U of E^* are orthogonal iff
$$\langle u^*, v \rangle = 0 \quad \forall u^* \in U \text{ and } v \in V.$$
The *orthogonal* V^0 of V is the subspace V^0 of E^* defined s.t. $V^0 = \{ u^* \in E^* \mid \langle u^*, v \rangle = 0 \quad \forall v \in V \}$ (also called annihilator of V)

The *orthogonal* U^0 of U is
$$U^0 = \{ v \in E \mid \langle u^*, v \rangle = 0 \quad \forall u^* \in U \}$$

Although we've discussed all of these in the context of general vector spaces, we can specialize to Euclidean spaces.

Def. 11.1 A Euclidean space is a real vector space E equipped vector spaces, we can specialize to ...

Def. 11.1 A Euclidean space is a real vector space E equipped with a symmetric bilinear form $\varphi : E \times E \to \mathbb{R}$ that is positive definite.

Notation: $\varphi(u,v) = u \cdot v = \langle u, v \rangle = (u \mid v)$

$\Phi(u) = \varphi(u,u) = \|u\|^2$.

Def. 11.2 Given a Euclidean space E, $u, v \in E$ are orthogonal/perpendicular if $u \cdot v = 0$. Given a family $(u_i)_{i \in I}$ of vectors in E, $(u_i)_{i \in I}$ is orthogonal if $u_i \cdot u_j = 0$ if $i \neq j$. $(u_i)_{i \in I}$ is orthonormal if in addition $\|u_i\| = u_i \cdot u_i = 1$ $\forall i \in I$. For any subset F of E, the set

$$F^{\perp} = \{ v \in E \mid u \cdot v = 0, \ \forall u \in F \}$$

of all vectors orthogonal to all vectors in F, is called the orthogonal complement of F.

Def. 11.3 For any $u \in E$, let $\varphi_u : E \to \mathbb{R}$ be the map defined s.t.

$$\varphi_u(v) = u \cdot v, \quad \forall v \in E.$$

Note that φ_u is a linear form in E^*.

Thm 11.1/11.6 The map $\flat : E \to E^*$ defined s.t.

$$\flat(u) = \varphi_u.$$

is linear and injective. When E is finite dim., $\flat$ is a canonical isomorphism.

proof. $\flat$ is linear because $\langle , \rangle$ is bilinear.

If $\varphi_u = \varphi_v$, then $\varphi_u(w) = \varphi_v(w)$ $\forall w \in E$

$\Rightarrow u \cdot w = v \cdot w$ $\forall w \in E$

$\Rightarrow (v - u) \cdot w = 0$

$\Rightarrow u = v$, because $\langle , \rangle$ is pos. def.

$\Rightarrow \flat$ is injective

If $\dim(E) = n < \infty$, $\dim(E) = \dim(E^*)$, so $\flat$ is bijective. $\boxed{}$

Notation: The inverse of the isomorphism $\flat : E \to E^*$ is denoted
$$\# : E^* \to E.$$

Corollary 11.1/11.7 If $\dim(E) = n < \infty$, every linear form $f \in E^*$ corresponds to a unique $u \in E$ s.t.
$$f(v) = u \cdot v, \quad \forall\, v \in E.$$
If f is not the 0-form, then $\operatorname{Ker} f$ is a hyperplane of all vectors orthogonal to u.

Note: $\flat$ is not surjective if E has infinite dim.
Can be salvaged by looking only at continuous linear maps in a Hilbert space (complete normed space)
(Riesz representation theorem)

Prop. 11.6/11.8 If $\dim(E) = n < \infty$, for every linear map $f : E \to E$, there is a unique linear map $f^* : E \to E$ s.t.
$$f^*(u) \cdot v = u \cdot f(v) \quad \forall\, u, v \in E.$$

proof. Suppose $a_1 \cdot v = u \cdot f(v)$ and $a_2 \cdot v = u \cdot f(v)$
$$\Rightarrow \quad a_1 \cdot v = a_2 \cdot v$$
$$\Rightarrow \quad (a_1 - a_2) \cdot v = 0$$
$$\Rightarrow \quad a_1 = a_2 \quad \text{or} \quad v = 0.$$
But since v is any element of V, $\Rightarrow$ $a_1 = a_2$.
This implies that $f^*(u) = a_1$ is uniquely defined, so we just need linearity.

Given $u_1, u_2 \in E$, $(u_1 + u_2) \cdot f(v) = u_1 \cdot f(v) + u_2 \cdot f(v)$, $\forall\, v \in E$.

and $(f^*(u_1) + f^*(u_2)) \cdot v = f^*(u_1) \cdot v + f^*(u_2 \cdot v)$ $\forall\, v \in E$.

By assumption, $f^*(u_1) \cdot v = u_1 \cdot f(v)$ and $f^*(u_2) \cdot v = u_2 \cdot f(v)$, $\forall\, v \in E$.

$$\Rightarrow (f^*(u_1) + f^*(u_2)) \cdot v = (u_1 + u_2) \cdot f(v) = f^*(u_1 + u_2) \cdot v, \quad \forall v \in E.$$

Since $\flat$ is bijective $\Rightarrow$ $f^*(u_1 + u_2) = f^*(u_1) + f^*(u_2)$.

Similary, $\forall v$, $f^*(\lambda u)\cdot v = \lambda u \cdot f(v)$

$$= \lambda(u \cdot f(v))$$
$$= \lambda(f^*(u)\cdot v)$$
$$= \lambda f^*(u)\cdot v$$
$$\Rightarrow \quad f^*(\lambda u) = \lambda f^*(u).$$

$\boxtimes$

Def. 11.4 If $\dim(E) = n < \infty$, $\forall$ linear map $f: E \to E$, the unique linear map $f^*: E \to E$ s.t.

$$f^*(u)\cdot v = u \cdot f(v), \quad \forall u, v \in E$$

is called the *adjoint* of f (w.r.t. the inner prod.).

Linear maps $f: E \to E$ s.t. $f = f^*$ are called *self-adjoint*.

Definition Given Euclidean spaces $(E, \langle\, ,\rangle_1)$ and $(F, \langle\, ,\rangle_2)$, and a linear map $f: E \to F$, we can similarly show that there exists a unique linear map $f^*: F \to E$ s.t.

$$\langle f(u), v\rangle_2 = \langle u, f^*(v)\rangle_1. \quad \forall u \in E,\ v \in F.$$

f^* is called the *adjoint* of f.

Properties

(1) $f^{**} = f$

(2) $(f+g)^* = f^* + g^*$

$(\lambda f)^* = \lambda f^*$

(3) $(g \circ f)^* = f^* \circ g^*$.

Prop. 11.7 / 11.9 Given any nontrivial Euclidean space E of finite dim. $n \geq 1$, $\exists$ an orthonormal basis $(u_1, \ldots, u_n)$ for E.

proof. Induction on n. Base case $n = 1$: take any $v \neq 0$ and let $u_1 = \frac{v}{\|v\|}$.

For $n \geq 2$, start with $u_1 = \frac{v}{\|v\|}$ for $v \neq 0$.

Let $H = \{u_1\}^\perp$, the hyperplane that is the orthogonal complement of $\{u_1\}$.

... φ is the linear form associated

Let $H = \{u_1\}^\perp$, the hyperplane that is the orthogonal complement of $\{u_1\}$.
$\dim(H) = n-1$ because $H = \mathrm{Ker}\,\varphi_{u_1}$, where φ_{u_1} is the linear form associated with u_1.

Thus, by the inductive hypo, H has an orthonormal basis $(u_2, \dots, u_n)$.
Then $(u_1, \dots, u_n)$ is an orthonormal basis of $E = H \oplus \mathbb{R} u_1$. ☐

Prop. 11.8/11.10 Given any nontrivial Euclidean space of finite dim $n \geq 1$, from any basis $(e_1, \dots, e_n)$ for E, we can construct an orthonormal basis $(u_1, \dots, u_n)$ s.t. $\forall k$, $\mathrm{span}\{e_1, \dots, e_k\} = \mathrm{span}\{u_1, \dots, u_k\}$.

proof sketch Let $v_k = e_k - \sum_{i=1}^{k-1} \varphi_{u_i}(e_k) u_i$, $u_k = \frac{v_k}{\|v_k\|}$.

We remove from e_k its projection onto $\mathrm{span}\{u_1, \dots, u_{k-1}\} = \mathrm{span}\{e_1, \dots, e_{k-1}\}$. ☐

This is called Gram-Schmidt orthonormalization.

Prop. 11.9/11.11 Given any nontrivial Euclidean space of finite dim $n \geq 1$, for any subspace F of dim. k, the orthogonal complement $F^\perp$ has dim $n-k$ and $E = F \oplus F^\perp$. Furthermore $F^{\perp\perp} = F$.

proof sketch Start with an orthonormal basis on F, extend to a basis on E, then orthonormalize it to get an orthonormal basis $(u_1, \dots, u_n)$ on E where $\mathrm{span}\{u_1, \dots, u_k\} = F$
$\mathrm{span}\{u_{k+1}, \dots, u_n\} = F^\perp$. ☐

Def. 11.5 Given any two nontrivial Euclidean spaces E and F of the same finite dim n, a function $f : E \to F$ is an orthogonal transformation, or linear isometry, if it is linear and $\|f(u)\| = \|u\|$ for all $u \in E$.

Prop. 11.10/11.12 Given any two nontrivial Euclidean spaces E and F of the same finite dim n, $\forall f : E \to F$, the following are equiv:

(1) f is a linear map and $\|f(u)\| = \|u\|$, $\forall u \in E$

the same finite dim n, $\forall f: E \to F$, the following are equiv:

(1) f is a linear map and $\|f(u)\| = \|u\|$, $\forall u \in E$

(2) $\|f(v) - f(u)\| = \|v - u\|$, $\forall u, v \in E$ and $f(0) = 0$.

(3) $f(u) \cdot f(v) = u \cdot v$ $\forall u, v \in E$.

Furthermore, such a map is bijective.

proof. (1)→(2): $\|f(v) - f(u)\| = \underbrace{\|f(v-u)\|}_{\text{linearity}} = \|v - u\|$.

(2)→(3): Claim: If $\|f(v) - f(u)\| = \|v - u\|$ $\forall u, v \in E$, then $\forall \tau \in E$,

Let's prove something slightly more general.

the function $g: E \to F$ defined by $g(u) = f(\tau + u) - f(\tau)$ $\forall u \in E$ is a linear map s.t. $g(0) = 0$ and $g(u) \cdot g(v) = u \cdot v$.

proof. Clearly, $g(0) = f(\tau) - f(\tau) = 0$.

$$\|g(v) - g(u)\| = \|f(\tau + v) - f(\tau) - f(\tau - u) + f(\tau)\|$$
$$= \|f(\tau + v) - f(\tau + u)\|$$
$$= \|\tau + v - (\tau + u)\| = \|v - u\| \quad \forall u, v \in E.$$

Also, setting $u = 0$, $\|g(v)\| = \|v\|$, so g preserves norms and distances.

Further, $\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 + \|v\|^2 + 2 u \cdot v$

$$\Rightarrow 2 u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2.$$

Thus $2 g(u) \cdot g(v) = \|g(u)\|^2 + \|g(v)\|^2 - \|g(u) - g(v)\|^2$

$$= \|u\|^2 + \|v\|^2 - \|u - v\|^2$$
$$= 2 u \cdot v$$

$$\Rightarrow g(u) \cdot g(v) = u \cdot v \quad \forall u, v \in E.$$

If we let $\tau = 0$, then $g = f$, showing that $f(u) \cdot f(v) = u \cdot v$.

(3)→(1): $f(u) \cdot f(u) = u \cdot u$

$$\Rightarrow \|f(u)\|^2 = \|u\|^2 \quad \forall u \in E.$$

$$\Rightarrow \|f(u)\| = \|u\|.$$

Now just need to show f is linear.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of E.

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Let $(e_1, \dots, e_n)$ be an orthonormal basis of E.

So $\quad u = \sum_{i=1}^{n} (u \cdot e_i) e_i = \sum_{i=1}^{n} u_i e_i \quad$ for $u \in E$.

Then $(f(e_1), \dots, f(e_n))$ is another orthonormal basis of E.

Then
$$f(u) = \sum_{i=1}^{n} \big(f(u) \cdot f(e_i) \big) f(e_i)$$
$$= \sum_{i=1}^{n} (u \cdot e_i)\, f(e_i) = \sum_{i=1}^{n} u_i f(e_i).$$

This proves that f is linear by using this basis.

Last: If $f(u) = f(v)$, then $f(v-u) = 0$, so $\|f(v-u)\| = 0$

$\Rightarrow \|v-u\| = 0 \quad \Rightarrow u = v.$

Thus, f is injective, and since $\dim(E) = \dim(F) = n$, f is bijective.

Prop. 11.11/11.13 Given any two nontrivial Euclidean spaces E and F of the same finite dim. n, $\forall f: E \to F$, if
$$\|f(v) - f(u)\| = \|v - u\| \quad \forall u, v \in E,$$
then f is an affine map, and its associated linear map g is an isometry.

proof. See Prop. 11.10/11.12 (2) $\to$ (3) proof.

Prop. 11.12/11.14 Let E be any Euclidean space of finite dim n, and let $f: E \to E$ be any linear map. Then

(1) f is an isometry iff $f \circ f^* = f^* \circ f = id$.

(2) For any orthonormal basis $(e_1, \dots, e_n)$ of E, if the matrix of f is A, then the matrix of f^* is A^T, and f is an isometry iff $AA^T = A^TA = I_n$.

iff columns of A form an orthonormal basis of $\mathbb{R}^n$

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iff rows of A form an orthonormal basis of $\mathbb{R}^n$.

proof. (1) $f(u)\cdot f(v) = u\cdot v$ by def. of isometry

$\Leftrightarrow f^*(f(u))\cdot v = u\cdot v$ by def. of adjoint

$\Leftrightarrow f^*(f(u)) = u$ because true $\forall v$.

$\Leftrightarrow f^*\circ f = id.$

But an endomorphism f in a finite-dim space with a left-inverse is an isomorphism, so $f^*\circ f = id.$

(2) If $(e_1, \dots, e_n)$ is an orthonormal basis of E, let $A = (a_{ij})$ be the matrix of f and $B = (b_{ij})$ be the matrix of f^*.

i.e., columns of A are the images of f under the basis.

$$A = \begin{pmatrix} f(e_1) & \cdots & f(e_n)\end{pmatrix} \qquad B = \begin{pmatrix} f^*(e_1) & \cdots & f^*(e_n)\end{pmatrix}$$

Let $w = w_1e_1 + \dots + w_ne_n \in E$. Then $w_k = w_i \cdot w$.

Then $b_{ji} = f^*(e_i)\cdot e_j = e_i \cdot f(e_j) = a_{ij}$

$$\Rightarrow B = A^T.$$

Then
$$A^TA = \begin{pmatrix} f(e_1)^T \\ \vdots \\ f(e_n)^T\end{pmatrix}\begin{pmatrix} f(e_1) & \cdots & f(e_n)\end{pmatrix} = \begin{pmatrix} f(e_1)\cdot f(e_1) & f(e_1)\cdot f(e_2) & \cdots & f(e_1)\cdot f(e_n) \\ \vdots & \ddots & & \vdots \\ f(e_n)\cdot f(e_1) & \cdots & \cdots & f(e_n)\cdot f(e_n)\end{pmatrix}$$

$\left(\text{But } (f(e_i))_{i\in I} \text{ form another orthonormal basis because } f \text{ preserves inner prod.}\right) = I_n.$

Similarly, $(AA^T)_{ij} = f^*(e_i)\, f^*(e_j) = \delta_{ij}$, so $AA^T = I_n$, and we also get columns of A^T are orthonormal. $\boxed{}$

Def. 11.6 A real $n\times n$ matrix is an *orthogonal matrix* if

$$AA^T = A^TA = I$$

Def. 11.6 A real $n \times n$ matrix is an orthogonal matrix if
$$A A^T = A^T A = I_n.$$
(Equivalent to $A^T = A^{-1}$)